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On 2-adic Cyclotomic Elements in K-theory and Étale Cohomology of the Ring of Integers

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In this paper we define 2-adic cyclotomic elements in K-theory and étale cohomology of the integers. We construct a comparison map which sends the 2-adic elements in K-theory onto 2-adic elements in cohomology. Using calculation of 2-adic K-theory of the integers due to Voevodsky, Rognes and Weibel, we show which part of the group $K_{2n-1}(\mathbb{Z}) \otimes \mathbb{Z}_2^{\wedge}$ for n odd, is described by the 2-adic cyclotomic elements. We compute explicitly some of the product maps in K-theory of $\mathbb{Z}$ at the prime 2. © 2000 Academic Press

0. INTRODUCTION

The main purpose of this paper is to give a detailed account of the 2-adic cyclotomic elements in the K-theory and étale cohomology of the ring of integers $\mathbb{Z}$. The cyclotomic elements for odd primes were introduced to K-theory by Soulé in the eighties ([So3], see also [So2]). There exist quite a few applications of the cyclotomic elements for odd primes, for instance the proof of the Tamagawa number conjecture for the Riemann zeta function (see [BK], p. 383) and the computation of p-adic regulator maps in K-theory (cf. [So3], [HW]). On the other hand, to the best of our knowledge, the 2-adic cyclotomic elements in K-theory and étale cohomology have

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not been discussed thoroughly in the published literature. There are some well known technical problems with the construction of 2-adic cyclotomic elements. Namely:

(a) a number field with at least one real imbedding has infinite 2-adic cohomological dimension

(b) there does not exist a well behaved Dwyer–Friedlander spectral sequence at the prime 2 for fields as in (a)

(c) there are problems with the product structure in K-theory with $\mathbb{Z}/2^k$ coefficients for $k \leqslant 3$

(d) there is no uniquely defined Bott element in K-theory with $\mathbb{Z}/2$ coefficients (cf. [W1]).

In this paper we manage to avoid the difficulties (a)–(d). Let us present the contents of the paper. In Sections 1 and 2 we define 2-adic cyclotomic elements. Section 3 contains some explicit computation of 2-adic cyclotomic elements in étale cohomology. The main result of this section is the following theorem.

THEOREM A [Th. 3.5]. *For any odd integer n, there is a natural isomorphism of $\mathbb{Z}_2^{\wedge}$-modules*

$$H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}]; \mathbb{Z}_2^{\wedge}(n)) \cong \mathscr{C}^{\text{ét}}(n-1) \oplus \mathbb{Z}/2,$$

where $\mathscr{C}^{\text{ét}}(n-1)$ is the group of 2-adic cyclotomic elements in $H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}]; \mathbb{Z}_2^{\wedge}(n))$. Moreover, the $\mathbb{Z}_2^{\wedge}$-module $\mathscr{C}^{\text{ét}}(n-1)$ is isomorphic to $\mathbb{Z}_2^{\wedge}$.

To get around problems (a) and (b) we introduce in Section 4 relative versions of Dwyer–Friedlander étale K-theory and étale cohomology. Using étale K-theory and cohomology we construct comparison maps of the following type.

THEOREM B [Th. 5.5]. *Let $X = \operatorname{spec} \mathbb{Z}[\tfrac{1}{2}]$ and $Y = \operatorname{spec} \mathbb{R}$, let n be an odd natural number and let k be a natural number. There exist natural maps*

$$K^{\text{ét}}_{2n-1}(X, Y; \mathbb{Z}/2^k) \to H^1_{\text{ét}}(X, Y; \mathbb{Z}/2^k(n))$$
$$\hat{K}^{\text{ét}}_{2n-1}(X, Y) \to H^1_{\text{ét}}(X, Y; \mathbb{Z}_2^{\wedge}(n)).$$

Composing the above maps with the Dwyer–Friedlander map we get natural homomorphisms

$$K_{2n-1}(X, Y; \mathbb{Z}/2^k) \to H^1_{\text{ét}}(X, Y; \mathbb{Z}/2^k(n))$$
$$K_{2n-1}(X, Y) \to H^1_{\text{cont}}(X, Y; \mathbb{Z}_2^{\wedge}(n)).$$

In Section 5 we address the question of which part of the group $K_{2n-1}(\mathbb{Z}) \otimes \mathbb{Z}_2^\wedge$ for n odd, is described by the 2-adic cyclotomic elements.

THEOREM C [Th. 5.10, Th. 5.14]. *Let n be an odd natural number. Let*

$$c_{n,1}: K_{2n-1}(X, Y; \mathbb{Z}_2^\wedge) \to H^1_{\text{ét}}(X, Y; \mathbb{Z}_2^\wedge(n))$$

denote the map induced by the third homomorphism of Theorem B upon taking inverse limit on coefficients. The map $c_{n,1}$ sends the group of 2-adic cyclotomic elements $\mathscr{C}(n-1)$ in K-theory isomorphically onto the group $\mathscr{C}^{\text{ét}}(n-1)$.

THEOREM D [Th. 5.14, Cor. 5.15].

(a) *For any odd integer n, there is a natural isomorphism of $\mathbb{Z}_2^\wedge$-modules*

$$K_{2n-1}(\mathbb{Z}) \otimes \mathbb{Z}_2^\wedge \cong \begin{cases} \mathscr{C}(n-1) & \text{if } n \equiv 3 \bmod 4 \\ \mathscr{C}(n-1) \oplus \mathbb{Z}/2 & \text{if } n \equiv 1 \bmod 4. \end{cases}$$

Moreover, the $\mathbb{Z}_2^\wedge$-module $\mathscr{C}(n-1)$ is isomorphic to $\mathbb{Z}_2^\wedge$.

(b) *For any n odd, there is a natural isomorphism*

$$K_{2n-1}(X, Y; \mathbb{Z}_2^\wedge) \cong \mathscr{C}(n-1).$$

Our proof of Theorem D relies on the calculation of the groups $K_*(\mathbb{Z}) \otimes \mathbb{Z}_2^\wedge$ by Rognes and Weibel cf. [RW] Theorem 0.6, which was made possible by the celebrated results of Voevodsky. (For the convenience of the reader we recall the results of the calculation in the beginning of Section 6.) Finally, in Section 6 we give applications of 2-adic cyclotomic elements to compute products in 2-adic K-theory of the integers.

THEOREM E [Th. 6.1]. *Assume that n and m are odd integers $\geqslant 3$.*

(a) *The product map*

$$\star: K_1(\mathbb{Z}) \otimes K_{2m-1}(\mathbb{Z} \to K_{2m}(\mathbb{Z})$$

sends the subgroup $\langle -1 \rangle \otimes \langle b_m \rangle$ to zero.

(b) *The 2-adic product map*

$$\star: K_{2n-1}(\mathbb{Z}) \otimes K_{2m-1}(\mathbb{Z}) \to K_{2(n+m-1)}(\mathbb{Z}) \otimes \mathbb{Z}_2^\wedge$$

sends the subgroup $\langle b_n \rangle \otimes \langle b_m \rangle$ to zero.

The element $b_n \in K_{2n-1}(\mathbb{Z})$ is such that $b_n \otimes 1$ is a generator of the free $\mathbb{Z}_2^\wedge$-module

$$\mathscr{C}(n-1) \subset K_{2n-1}(\mathbb{Z}) \otimes \mathbb{Z}_2^\wedge$$

1. 2-ADIC CYCLOTOMIC ELEMENTS IN $K_{2n-1}(\mathbb{Z}[\frac{1}{2}]) \otimes \mathbb{Z}_2^\wedge$

DEFINITION 1.1. For a commutative ring with identity R and $i \geqslant 0$ we denote by

$$K_i(R; \mathbb{Z}_2^\wedge) = \varprojlim K_i(R; \mathbb{Z}/2^m)$$

the 2-adic K-groups of R (cf. [So1] and [So3]), where the inverse limit is taken over reduction maps

$$r_m\colon K_i(R; \mathbb{Z}/2^{m+1}) \to K_i(R; \mathbb{Z}/2^m).$$

Observe that if $R = \mathcal{O}_F$ is the ring of integers in a number field F, then $K_i(\mathcal{O}_F; \mathbb{Z}_2^\wedge) \cong K_i(\mathcal{O}_F) \otimes \mathbb{Z}_2^\wedge$. In addition, if $R = F$, then $K_{2n-1}(F; \mathbb{Z}_2^\wedge) \cong K_{2n-1}(F) \otimes \mathbb{Z}_2^\wedge$ for $n > 1$.

In this section, we discuss a 2-adic version of cyclotomic elements (see [So2], Sections 4.3 and 4.4, p. 239–240). We construct the subgroup of cyclotomic elements in the 2-adic K-groups of $\mathbb{Z}[\frac{1}{2}]$

$$K_{2n-1}(\mathbb{Z}) \otimes \mathbb{Z}_2^\wedge \cong K_{2n-1}(\mathbb{Z}[\tfrac{1}{2}]) \otimes \mathbb{Z}_2^\wedge \cong \varprojlim K_{2n-1}(\mathbb{Z}[\tfrac{1}{2}]; \mathbb{Z}/2^m).$$

We assume that n is an odd integer and use the following notation:

$\xi_{2^m} =$ $\exp\left(\frac{2\pi i}{2^m}\right)$ is a primitive root of unity of order 2^m, where m is a positive integer,

$E_m =$ $\mathbb{Q}(\xi_{2^m})$ is the corresponding cyclotomic field,

$G_m =$ $Gal(E_m/\mathbb{Q})$ is the Galois group over $\mathbb{Q}$.

$R_m =$ $\mathbb{Z}[\xi_{2^m}]$ is the ring of integers of E_m,

$R'_m =$ $\mathbb{Z}[\frac{1}{2}, \xi_{2^m}]$,

$\mathbb{Z}/2^m(n)$ is the nth Tate twist of the sheaf of roots of unity on Spec R'_k for $k \geqslant 1$,

$\langle g_1, g_2, ..., g_r \rangle$ denotes the subgroup generated by elements $g_1, g_2, ..., g_r$ of a given abelian group G,

$C_m^+ =$ $\left\langle -1, \xi_{2^m}^{(1-a)/2} \dfrac{1-\xi_{2^m}^a}{1-\xi_{2^m}}, 1 < a < 2^{m-1}, a \text{ odd} \right\rangle$ is the group of real cyclotomic units in $E_m^\times$ (see [Wa], p. 145),

C_m denotes the subgroup of $R_m^\times$ generated by C_m^+ and ξ_{2^m},

C'_m is the subgroup of $R_m'^\times$ generated by C_m and $1-\xi_{2^m}^a$ for $1 \leqslant a < 2^m$ and a odd.

M_G denotes the coinvariants of a module M with an action of a group G.

Remark 1.2. It is easy to show that C'_m is generated by ξ_{2^m} and $1-\xi_{2^m}^a$ for $1 \leqslant a < 2^m$ and a odd. Observe that the group C'_m is also generated by $1-\xi_{2^m}^a$ and $\xi_{2^m}(1-\xi_{2^m}^a)$ for $1 \leqslant a < 2^m$ and a odd.

EXAMPLE 1.3. The group C'_2 is generated by $1-i$ and $1+i$ because $i(1-i) = 1+i$ and $i(1+i) = -(1-i)$.

Remark 1.4. We leave to the reader to check that

$$N_{E_m/\mathbb{Q}}(\xi_{2^m}^b(1-\xi_{2^m}^a)) = 2$$

for all $a, b \in \mathbb{Z}$ with $1 \leqslant a < 2^m$ and a odd, where $N_{E_m/\mathbb{Q}}$ denotes the norm map down to $\mathbb{Q}$.

Let us assume for a moment that $m \geqslant 2$ and consider the map

$$\alpha_m \colon C'_m \otimes \mathbb{Z}/2^m(n-1) \to K_{2n-1}(R'_m; \mathbb{Z}/2^m)$$

given by the formula

$$\alpha_m(c \otimes \xi_{2^m}^{\otimes (n-1)}) = c \star \beta_m^{\star (n-1)},$$

where $c \in C'_m \subseteq K_1(R'_m)$ and $\beta_m = \beta(\xi_{2^m}) \in K_2(R'_m; \mathbb{Z}/2^m)$ is the Bott element (see Definition 2.7.2 of [W1]). The symbol $\star$ denotes the product in algebraic K-theory. We shall also use the transfer map in K-theory (see [Q1], Section 4)

$$Tr_{E_m/\mathbb{Q}} \colon K_{2n-1}(R'_m; \mathbb{Z}/2^m) \to K_{2n-1}(\mathbb{Z}[\tfrac{1}{2}]; \mathbb{Z}/2^m).$$

DEFINITION 1.5. Let n be any odd integer $\geqslant 3$.

(a) Let $\mathscr{C}_m(n-1) \subseteq K_{2n-1}(\mathbb{Z}[\frac{1}{2}]; \mathbb{Z}/2^m)$ denote the image of the composition $Tr_{E_m/\mathbb{Q}} \circ \alpha_m$ for $m \geqslant 2$.

(b) We define $\mathscr{C}_1(n-1)\subseteq K_{2n-1}(\mathbb{Z}[\frac{1}{2}];\mathbb{Z}/2)$ to be the group $r_1\mathscr{C}_2(n-1)$, where

$$r_1\colon K_{2n-1}(\mathbb{Z}[\tfrac{1}{2}];\mathbb{Z}/4)\to K_{2n-1}(\mathbb{Z}[\tfrac{1}{2}];\mathbb{Z}/2)$$

is the reduction of coefficients on K-groups.

(c) Let $\mathscr{C}(n-1)=\varprojlim\mathscr{C}_m(n-1)$ which is a subgroup of

$$\varprojlim K_{2n-1}(\mathbb{Z}[\tfrac{1}{2}];\mathbb{Z}/2^m)\cong K_{2n-1}(\mathbb{Z}[\tfrac{1}{2}])\otimes\mathbb{Z}_2^\wedge\cong K_{2n-1}(\mathbb{Z})\otimes\mathbb{Z}_2^\wedge.$$

We call $\mathscr{C}(n-1)$ the group of 2-adic cyclotomic elements in $K_{2n-1}(\mathbb{Z})\otimes\mathbb{Z}_2^\wedge$.

(d) In the same way we define: $\widetilde{\mathscr{C}}_m(n-1)\subseteq K_{2n-1}(\mathbb{Z}[i,\frac{1}{2}];\mathbb{Z}/2^m)$, to be the image of the composition of maps $Tr_{E_m/E_2}\circ\alpha_m$ for $m\geqslant 2$ and $\widetilde{\mathscr{C}}_1(n-1)\subseteq K_{2n-1}(\mathbb{Z}[i,\frac{1}{2}];\mathbb{Z}/2)$ to be the image of $\widetilde{\mathscr{C}}_2(n-1)$ via the reduction of coefficients. We put $\widetilde{\mathscr{C}}(n-1)=\varprojlim\widetilde{\mathscr{C}}_m(n-1)\subseteq K_{2n-1}(\mathbb{Z}[i,\frac{1}{2}];\mathbb{Z}_2^\wedge)=K_{2n-1}(\mathbb{Z}[i,\frac{1}{2}])\otimes\mathbb{Z}_2^\wedge$.

LEMMA 1.6. *For any $n>1$ we have the following equality*

$$Tr_{E_2/\mathbb{Q}}(\widetilde{\mathscr{C}}(n-1))=\mathscr{C}(n-1).$$

Proof. It is obvious from Definition 1.5(a), (d) that

$$Tr_{E_2/\mathbb{Q}}(\widetilde{\mathscr{C}}_m(n-1)=\mathscr{C}_m(n-1)$$

for $m\geqslant 2$. From Definition 1.5(b) and the above equality for $m=2$ we get

$$Tr_{E_2/\mathbb{Q}}(\widetilde{\mathscr{C}}_1(n-1))=\mathscr{C}_1(n-1).$$

The groups $\mathscr{C}_m(n-1)$ and $\widetilde{\mathscr{C}}_m(n-1)$ are finite for all m. Hence, taking inverse limit over m we obtain the claim of the lemma. ∎

2. 2-ADIC CYCLOTOMIC ELEMENTS IN $H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2}];\mathbb{Z}_2^\wedge(n))$

In this section we define 2-adic cyclotomic elements inside the étale cohomology group

$$H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}];\mathbb{Z}_2^\wedge(n))\cong\varprojlim H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}];\mathbb{Z}/2^m(n)).$$

Since $\mathrm{Pic}(R'_m)_2$ is a trivial group by [We], Satz C, p. 244, it follows from the Kummer exact sequence that we have the isomorphism

$$\begin{aligned}H^1_{\text{ét}}(R'_m;\mathbb{Z}/2^m(n))&\cong H^1_{\text{ét}}(R'_m;\mathbb{Z}/2^m(1))\otimes\mathbb{Z}/2^m(n-1)\\&\cong R'^{\times}_m\otimes\mathbb{Z}/2^m(n-1).\end{aligned}$$

Hence we get the composition of maps

$$\alpha_m^{\text{ét}}\colon C'_m \otimes \mathbb{Z}/2^m(n-1) \to R'^{\times}_m \otimes \mathbb{Z}/2^m(n-1) \xrightarrow{\cong} H^1_{\text{ét}}(R'_m; \mathbb{Z}/2^m(n)),$$

where the first map is induced by the natural injection. Note that we defined the map $\alpha_m^{\text{ét}}$ and the map α_m of the previous section only for $m \geqslant 2$.

LEMMA 2.1. *The map $\alpha_m^{\text{ét}}$ is an isomorphism for all $m \geqslant 2$.*

Proof. It is sufficient to prove that $R'^{\times}_m/2^m \cong C'_m/2^m$. It follows from the commutative diagram of embeddings of groups

$$\begin{array}{ccc} WU_m & \longrightarrow & R_m^{\times} \\ \uparrow & & \uparrow \\ WC_m^{+} & \longrightarrow & C_m, \end{array}$$

where $W = \langle \xi_{2^m} \rangle$ and U_m is the group of units of the maximal real subfield of E_m, that the index $[R_m^{\times} : C_m]$ divides the product of indices $[R_m^{\times} : WU_m]$ $[WU_m : WC_m^{+}]$. Note that $WC_m^{+} = C_m$. By [Wa], Corollary 4.13, p. 39 and Theorem 8.2, p. 145, we know that $[R_m^{\times} : WU_m] = 1$ and that $[WU_m : WC_m^{+}] = [U_m : C_m^{+}]$ is the class number which is odd (see [We], Satz C, p. 244). Thus, we obtain $R_m^{\times}/2^m \cong C_m/2^m$ and, by the definition given at the beginning of section 1, $R'^{\times}_m/2^m \cong C'_m/2^m$. ∎

DEFINITION 2.2. Let n be any odd integer $\geqslant 3$.

(a) If $m \geqslant 2$, we define $\mathscr{C}_m^{\text{ét}}(n-1)$ to be the image of the composition

$$C'_m \otimes \mathbb{Z}/2^m(n-1) \xrightarrow[\cong]{\alpha_m^{\text{ét}}} H^1_{\text{ét}}(R'_m; \mathbb{Z}/2^m(n)) \xrightarrow{Tr_{E_m/\mathbb{Q}}} H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}]; \mathbb{Z}/2^m(n)),$$

where $Tr_{E_m/\mathbb{Q}}$ denotes the transfer map in étale cohomology.

(b) For $m = 1$ we put $\mathscr{C}_1^{\text{ét}}(n-1)$ to be the image of $\mathscr{C}_2^{\text{ét}}(n-1)$ via the reduction of coefficients

$$r_1\colon H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}]; \mathbb{Z}/4(n)) \to H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}]; \mathbb{Z}/2(n)).$$

(c) Taking inverse limits with respect to m above we obtain a subgroup $\mathscr{C}^{\text{ét}}(n-1)$ of

$$H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}]; \hat{\mathbb{Z}}_2(n)) \cong \varprojlim H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}]; \mathbb{Z}/2^m(n))$$

which we call the subgroup of étale 2-adic cyclotomic elements.

(d) In the same way we define: $\widetilde{\mathscr{C}}_m^{\text{ét}}(n-1) \subseteq H^1_{\text{ét}}(\mathbb{Z}[i, \frac{1}{2}]; \mathbb{Z}/2^m(n))$ to be the image of the composition of maps $Tr_{E_m/E_2} \circ \alpha_m$ for $m \geqslant 2$ and $\widetilde{\mathscr{C}}_1^{\text{ét}}(n-1) \subseteq H^1_{\text{ét}}(\mathbb{Z}[i, \frac{1}{2}]; \mathbb{Z}/2(n))$ to be the image of $\widetilde{\mathscr{C}}_2^{\text{ét}}(n-1)$ via the reduction of coefficients. We put $\widetilde{\mathscr{C}}^{\text{ét}}(n-1) = \varprojlim \widetilde{\mathscr{C}}_m^{\text{ét}}(n-1) \subseteq H^1_{\text{ét}}(\mathbb{Z}[i, \frac{1}{2}]; \mathbb{Z}_2^{\wedge}(n))$.

Lemma 2.3. *For any $n > 1$ we have the following equality*

$$Tr_{E_2/\mathbb{Q}}(\widetilde{\mathscr{C}}^{\text{ét}}(n-1)) = \mathscr{C}^{\text{ét}}(n-1).$$

Proof. The argument is similar to the proof of Lemma 1.6. ∎

Lemma 2.4. *For any $n > 1$ the Dwyer–Friedlander map $K_{2n-1}(\mathbb{Z}[i, \frac{1}{2}]; \mathbb{Z}_2^{\wedge}) \to H^1_{\text{ét}}(\mathbb{Z}[i, \frac{1}{2}]; \mathbb{Z}_2^{\wedge}(n))$, (see [DF], p. 276) induces the surjection*

$$\widetilde{\mathscr{C}}(n-1) \to \widetilde{\mathscr{C}}^{\text{ét}}(n-1).$$

Proof. The Dwyer–Friedlander map for the ring R'_m and for finite coefficients $\mathbb{Z}/2^m$, where $m \geqslant 2$ is a ring homomorphism which commutes with transfer maps Tr_{E_m/E_2}, (see [DF], Prop. 4.4 and Th. 6.4). For this reason, if $m \geqslant 2$, the Dwyer–Friedlander map induces a surjection

$$\widetilde{\mathscr{C}}_m(n-1) \to \widetilde{\mathscr{C}}_m^{\text{ét}}(n-1).$$

By Definitions 1.5 and 2.2 and by the commutativity of the Dwyer–Friedlander map with the reduction of coefficients from $\mathbb{Z}/4$ to $\mathbb{Z}/2$ we get the surjection

$$\widetilde{\mathscr{C}}_1(n-1) \to \widetilde{\mathscr{C}}_1^{\text{ét}}(n-1).$$

The groups $\widetilde{\mathscr{C}}_m(n-1)$ and $\widetilde{\mathscr{C}}_m^{\text{ét}}(n-1)$ are finite for all m. Hence, taking inverse limit over m we obtain the claim of the lemma. ∎

3. COMPUTATIONS IN 2-ADIC ÉTALE COHOMOLOGY

Observe that for $m \geqslant k \geqslant 2$ the norm map $N_{E_m/E_k}\colon C'_m \to C'_k$ is surjective by Remark 1.2. For $m \geqslant k > 2$, we have the following commutative diagram with exact rows which comes from the naturality of the Bockstein long exact sequence in étale cohomology:

$$\begin{tikzcd}
H^1_{\text{ét}}(R'_m; \mathbb{Z}/2^{k-1}(n)) \arrow[r] \arrow[d, "Tr_{E_m/E_2}"] & H^1_{\text{ét}}(R'_m; \mathbb{Z}/2^k(n)) \arrow[r, "r'"] \arrow[d, "Tr_{E_m/E_2}"] & H^1_{\text{ét}}(R'_m; \mathbb{Z}/2(n)) \arrow[d, "Tr_{E_m/E_2}"] \\
H^1_{\text{ét}}(R'_2; \mathbb{Z}/2^{k-1}(n)) \arrow[r] & H^1_{\text{ét}}(R'_2; \mathbb{Z}/2^k(n)) \arrow[r] & H^1_{\text{ét}}(R'_2; \mathbb{Z}/2(n)).
\end{tikzcd} \tag{3.1}$$

For $m \geqslant k \geqslant 2$, in a similar way to the proof of Lemma 2.1 we obtain the isomorphism

$$H^1_{\text{ét}}(R'_m; \mathbb{Z}/2^k(n)) \cong C'_m \otimes \mathbb{Z}/2^k(n-1)\,. \tag{3.2}$$

By the projection formula in étale cohomology (see [So1], Lemma 6) the right vertical map in Diagram (3.1) is surjective. In addition, the reduction of coefficients map r' in the top exact sequence is surjective. By induction over k we can assume that the left vertical map is surjective. Hence, it follows that the middle vertical map in (3.1) is surjective (see Diagram 2.6 of [BG]). Let us put

$$B_m = \begin{cases} r_1(\alpha_2^{\text{ét}}(C'_2 \otimes \mathbb{Z}/4(n-1))) & \text{if} \quad m=1, \\ Tr_{E_m/E_2}(\alpha_m^{\text{ét}}(C'_m \otimes \mathbb{Z}/2^m(n-1))) & \text{if} \quad m \geqslant 2, \end{cases} \tag{3.3}$$

where $r_1 \colon H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2}, i]; \mathbb{Z}/4(n)) \to H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2}, i]; \mathbb{Z}/2(n))$ is the reduction of coefficients and $i = \xi_4$. From the surjectivity of the transfer maps in Diagram (3.1) we obtain the isomorphism

$$H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}, i]; \mathbb{Z}_2^\wedge(n)) \cong \varprojlim H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}, i]; \mathbb{Z}/2^m(n)) \cong \varprojlim B_m, \tag{3.4}$$

where the inverse limit is taken with respect to the reduction of coefficients in étale cohomology.

Lemma 3.1. *For any $k \geqslant 1$, $m \geqslant 1$ and n odd, $H^1_{\text{ét}}(R'_k; \mathbb{Z}/2^m(n)) \cong H^1_{\text{ét}}(R'_k; \mathbb{Z}_2^\wedge(n))/2^m$.*

Proof. This is an exercise on long exact sequences in cohomology which are associated with the sequence of Galois modules

$$0 \to \mathbb{Z}_2^\wedge(n) \xrightarrow{2^m} \mathbb{Z}_2^\wedge(n) \to \mathbb{Z}/2^m(n) \to 0$$

and the sequence of sheaves

$$0 \to \mathbb{Z}/2(1) \to \mathbb{G}_{\mathrm{m}} \xrightarrow{2} \mathbb{G}_{\mathrm{m}} \to 0,$$

where $\mathbb{G}_{\mathrm{m}}$ is the multiplicative group scheme on Spec R'_k. The sequence of Galois modules induces a long exact sequence in étale cohomology, which gives the short exact sequence

$$\begin{aligned} 0 \to H^1_{\text{ét}}(R'_k; \mathbb{Z}_2^\wedge(n))/2^m &\to H^1_{\text{ét}}(R'_k; \mathbb{Z}/2^m(n)) \\ &\to H^2_{\text{ét}}(R'_k; \mathbb{Z}_2^\wedge(n))[2^m] \to 0\,. \end{aligned}$$

Here, $H^2_{\text{ét}}(R'_k; \mathbb{Z}_2^\wedge(n))[2^m]$ denotes the subgroup $\{x \in H^2_{\text{ét}}(R'_k;\ \mathbb{Z}_2^\wedge(n)) \mid 2^m x = 0\}$. We want to prove that the group $H^2_{\text{ét}}(R'_k; \mathbb{Z}_2^\wedge(n))[2^m]$ vanishes. Since the $\mathbb{Z}_2^\wedge$-module $H^2_{\text{ét}}(R'_k; \mathbb{Z}_2^\wedge(n))$ is finitely generated, we see that

$H^2_{\text{ét}}(R'_k; \mathbb{Z}_2^\wedge(n)) = 0$ if and only if $H^2_{\text{ét}}(R'_k; \mathbb{Z}_2^\wedge(n))/2 = 0$. The same long exact sequence in étale cohomology with $m = 1$ gives the short exact sequence

$$0 \to H^2_{\text{ét}}(R'_k; \mathbb{Z}_2^\wedge(n))/2 \to H^2_{\text{ét}}(R'_k; \mathbb{Z}/2(n)) \to H^3_{\text{ét}}(R'_k; \mathbb{Z}_2^\wedge(n))[2] \to 0.$$

In addition, by triviality of the Galois action on $\mathbb{Z}/2(1)$, we get the isomorphism

$$H^2_{\text{ét}}(R'_k; \mathbb{Z}/2(n)) \cong H^2_{\text{ét}}(R'_k; \mathbb{Z}/2(1)) \otimes \mathbb{Z}(n-1).$$

Now consider the long exact sequence in cohomology which is induced by the above exact sequence of sheaves: it gives the short exact sequence

$$0 \to \operatorname{Pic}(R'_k)/2 \to H^2_{\text{ét}}(R'_k; \mathbb{Z}/2(1)) \to \operatorname{Br}(R'_k)[2] \to 0,$$

since $H^1_{\text{ét}}(R'_k; \mathbb{G}_{\mathrm{m}}) = \operatorname{Pic}(R'_k)$ and $H^2_{\text{ét}}(R'_k; \mathbb{G}_{\mathrm{m}}) = \operatorname{Br}(R'_k)$ by [M1], p. 107 and p. 109. We know from [We], Satz C, p. 244, that $\operatorname{Pic}(R'_k)/2 = 0$. On the other hand, it follows from class field theory (see Example 2.22(f), p. 109 of [M1]) that:

$$\operatorname{Br}(R'_k) \text{ is } \mathbb{Z}/2 \text{ if } k = 1 \text{ and vanishes for } k \geqslant 2,$$

$$H^3_{\text{ét}}(R'_k; \mathbb{Z}/2(n)) \text{ is } \mathbb{Z}/2 \text{ if } k = 1 \text{ and vanishes for } k \geqslant 2.$$

This implies that $H^2_{\text{ét}}(R'_k; \mathbb{Z}/2(n)) \cong H^3_{\text{ét}}(R'_k; \mathbb{Z}_2^\wedge(n))[2]$. Consequently, we get $H^2_{\text{ét}}(R'_k; \mathbb{Z}_2^\wedge(n))/2 = 0$ and may deduce the lemma. ∎

By Theorem 10 of [W2] and Theorem 1(i) of [K], we know that

$$H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}, i]; \mathbb{Z}_2^\wedge(n)) \cong \mathbb{Z}_2^\wedge \oplus \mathbb{Z}/4\,.$$

Hence, by Lemma 3.1 applied to the case $k = m = 2$ we obtain:

$$H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}, i]; \mathbb{Z}/4(n)) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/4\,. \tag{3.5}$$

On the other hand by (3.2) the map $\alpha_2^{\text{ét}}$ gives an isomorphism

$$H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}, i]; \mathbb{Z}/4(n)) \cong C'_2 \otimes \mathbb{Z}/4(n-1)\,. \tag{3.6}$$

According to Example 1.3, the group C'_2 has two generators: $1 - i$ and $1 + i$. The module $\mathbb{Z}/4(n-1)$ has a canonical generator $\xi_4^{\otimes(n-1)}$. Hence, using the isomorphism (3.6) we see that the elements

$$(1 - i) \otimes \xi_4^{\otimes(n-1)} \qquad \text{and} \qquad (1 + i) \otimes \xi_4^{\otimes(n-1)}$$

generate $H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2}, i]; \mathbb{Z}/4(n))$. We will use these elements below to construct explicitly a generator of the $\mathbb{Z}_2^\wedge$-module $H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2}]; \mathbb{Z}_2^\wedge(n))$. For $m \geqslant 2$ we check directly that

$$\begin{aligned} N_{E_m/E_2}(1-\xi_{2^m}) &= 1-i \\ N_{E_m/E_2}(1-\xi_{2^m}^{2^m-1}) &= 1+i. \end{aligned} \tag{3.7}$$

Hence, by the definition of the inverse system (3.3) we obtain the two elements

$$e_1 = (b_m^{(1)}), \qquad e_2 = (b_m^{(2)}) \in \varprojlim B_m \cong H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}, i]; \mathbb{Z}_2^\wedge(n)),$$

where $b_m^{(1)}$ and $b_m^{(2)}$ are the elements of B_m defined as follows:

$$\begin{aligned} b_m^{(1)} &= \begin{cases} r_1(\alpha_2^{\text{ét}}((1-i)\otimes \xi_4^{\otimes(n-1)})) & \text{if} \quad m=1, \\ Tr_{E_m/E_2}(\alpha_m^{\text{ét}}((1-\xi_{2^m})\otimes \xi_{2^m}^{\otimes(n-1)})) & \text{if} \quad m\geqslant 2, \end{cases} \\ b_m^{(2)} &= \begin{cases} r_1(\alpha_2^{\text{ét}}((1+i)\otimes \xi_4^{\otimes(n-1)})) & \text{if} \quad m=1, \\ Tr_{E_m/E_2}(\alpha_m^{\text{ét}}((1-\xi_{2^m}^{2^m-1})\otimes \xi_{2^m}^{\otimes(n-1)})) & \text{if} \quad m\geqslant 2. \end{cases} \end{aligned} \tag{3.8}$$

LEMMA 3.2. *The elements e_1 and e_2 are nontorsion elements in $H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2}, i]; \mathbb{Z}_2^\wedge(n))$.*

Proof. If both e_1 and e_2 were torsion elements, then their images in the group (see Lemma 3.1)

$$H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}, i]; \mathbb{Z}/4(n)) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/4$$

under the reduction of coefficients mod 4 would be contained in the same $\mathbb{Z}/4$ summand of that group. This is impossible because the images of e_1 and e_2 generate the group $H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2}, i], \mathbb{Z}/4(n))$ by construction. It follows that at least one of them is nontorsion. On the other hand the complex conjugation τ acts on the group $H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2}, i]; \mathbb{Z}_2^\wedge(n))$. The automorphism τ sends e_1 to e_2 and e_2 to e_1. Indeed,

$$b_1^{(1)\,\tau} = r_1(\alpha_2^{\text{ét}}((1-i)^{\tau(-1)^{n-1}} \otimes \xi_4^{\otimes(n-1)})) = r_1(\alpha_2^{\text{ét}}((1+i)\otimes \xi_4^{\otimes(n-1)})) = b_1^{(2)}$$

because $(1-i)^\tau = 1+i$ and n is odd by assumption. For $m \geqslant 2$ we have

$$\begin{aligned} b_m^{(1)\,\tau} &= Tr_{E_m/E_2}(\alpha_m^{\text{ét}}((1-\xi_{2^m})^{\tau(-1)^{n-1}} \otimes \xi_{2^m}^{\otimes(n-1)})) \\ &= Tr_{E_m/E_2}(\alpha_m^{\text{ét}}((1-\xi_{2^m}^{2^m-1})\otimes \xi_{2^m}^{\otimes(n-1)})) = b_m^{(2)} \end{aligned}$$

because $(1-\xi_{2^m})^\tau = 1-\xi_{2^m}^{2^m-1}$ and n is odd by assumption. Hence, we obtain that $e_1^\tau = e_2$. Applying τ to the last equality we get $e_2^\tau = e_1$ because $\tau^2 = 1$. It follows that both elements must be nontorsion. ∎

Observe that by the isomorphism (3.2) and Definition 2.2 we have:

$$\mathscr{C}^{\text{ét}}(n-1) \cong \text{image}(H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}, i]; \mathbb{Z}_2^{\wedge}(n)) \xrightarrow{Tr_{E_2/E_1}} H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}]; \mathbb{Z}_2^{\wedge}(n))).$$

Recall that for n odd,

$$H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}]; \mathbb{Z}_2^{\wedge}(n)) \cong \mathbb{Z}_2^{\wedge} \oplus \mathbb{Z}/2$$

by Theorem 1(i) of [K] and Table 1 of [W2], and consider the surjective map

$$\phi\colon H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}]; \mathbb{Z}_2^{\wedge}(n)) \to H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}]; \mathbb{Z}/2(n)) \cong \langle -1, 2\rangle \otimes \mathbb{Z}/2(n-1),$$

where $\langle -1, 2\rangle = \mathbb{Z}[\frac{1}{2}]^{\times}/2$.

LEMMA 3.3. *The generator of the* 2-*torsion subgroup of* $H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2}]; \mathbb{Z}_2^{\wedge}(n))$ *is sent by the map* ϕ *to* $(-1) \otimes \xi_2^{\otimes (n-1)}$.

Proof. Let $S_0 = \{2\}$ be the set containing only the prime number 2, $\mathbb{Q}_{S_0}$ the maximal extension of $\mathbb{Q}$ unramified outside 2 and $G_{S_0} = G(\mathbb{Q}_{S_0}/\mathbb{Q})$. We know that

$$H^*_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}]; \mathscr{F}) \cong H^*(G_{S_0}; \mathscr{F})$$

for $\mathscr{F} = \mathbb{Z}/2^m(n)$ or $\mathscr{F} = \mathbb{Z}_2^{\wedge}(n)$ (see [M2], Proposition 2.9, p. 209). We have the commutative diagram

$$\begin{tikzcd} H^0(G_{S_0}; \mathbb{Z}/2(n)) \arrow[r, "\partial"] \arrow[d, "="] & H^1(G_{S_0}; \mathbb{Z}/2(n)) \arrow[d, "="] \\ \mathbb{Z}/2(n) \arrow[r] & \mathrm{Hom}(G(\mathbb{Q}(i, \sqrt{2})/\mathbb{Q}), \mathbb{Z}/2(n)), \end{tikzcd} \tag{3.9}$$

where the upper horizontal arrow is the connecting homomorphism in the long cohomology exact sequence associated with the exact sequence of G_{S_0}-modules

$$0 \to \mathbb{Z}/2(n) \to \mathbb{Z}/4(n) \to \mathbb{Z}/2(n) \to 0.$$

It is easy to observe that ∂ is an embedding. The Galois group $G(\mathbb{Q}(i, \sqrt{2})/\mathbb{Q}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ is generated by the complex conjugation τ and the automorphism σ such that $\sigma(i) = i$ and $\sigma(\sqrt{2}) = -\sqrt{2}$. Observe that the lower horizontal arrow in Diagram (3.9) sends the generator $\xi_2^{\otimes n}$ of

$\mathbb{Z}/2(n)$ to a homomorphism $f\colon G(\mathbb{Q}(i,\sqrt{2})/\mathbb{Q}) \to \mathbb{Z}/2(n)$ such that $f(\tau)=\xi_2^{\otimes n}$ and $f(\sigma)=1^{\otimes n}$. Let $\mathcal{O}_{S_0}$ denote the ring of integers in $\mathbb{Q}_{S_0}$ and consider the following commutative diagram:

$$\begin{array}{ccc} H^0(G_{S_0};\mathcal{O}_{S_0}^\times)/2 & \xrightarrow[\cong]{\partial'} & H^1(G_{S_0};\mathbb{Z}/2(1)) \\ \downarrow{=} & & \downarrow{=} \\ \langle -1,2\rangle & \xrightarrow{\cong} & \mathrm{Hom}(G(\mathbb{Q}(i,\sqrt{2})/\mathbb{Q}),\mathbb{Z}/2(1)). \end{array} \tag{3.10}$$

The upper horizontal arrow ∂' in Diagram (3.10) is the connecting homomorphism in the long cohomology exact sequence associated with the exact sequence

$$0 \to \mathbb{Z}/2(1) \to \mathcal{O}_{S_0}^\times \xrightarrow{2} \mathcal{O}_{S_0}^\times \to 0.$$

Tensoring the inverse map ∂'^{-1} with $\mathbb{Z}/2(n-1)$ we get the isomorphism

$$H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}];\mathbb{Z}/2(n)) \cong \langle -1,2\rangle \otimes \mathbb{Z}/2(n-1) \tag{3.11}$$

which we constantly use. Hence, the isomorphism (3.11) is just the inverse of the lower horizontal arrow in Diagram (3.10) tensored with $\mathbb{Z}/2(n-1)$ which we can compute explicitly. We are actually only interested in the image of (-1) tensored with $\xi_2^{\otimes(n-1)}$. We observe that $(-1)\otimes\xi_2^{\otimes(n-1)}$ is sent to a homomorphism f such that $f(\tau)=\xi_2^{\otimes n}$ and $f(\sigma)=1\otimes\xi_2^{\otimes(n-1)}=1^{\otimes n}$. Consider the commutative diagram

$$\begin{array}{ccc} H^0_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}];\mathbb{Z}/2(n)) & \xrightarrow{\partial} & H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}];\mathbb{Z}/2^m(n)) \\ \downarrow{=} & & \downarrow \\ H^0_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}];\mathbb{Z}/2(n)) & \xrightarrow{\partial} & H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}];\mathbb{Z}/2^{m-1}(n)), \end{array} \tag{3.12}$$

where the horizontal arrows are the connecting homomorphisms in the long exact sequences associated with the exact sequences

$$0 \to \mathbb{Z}/2^{j-1}(n) \to \mathbb{Z}/2^j(n) \to \mathbb{Z}/2(n) \to 0$$

for $j=m,\ m+1$. The horizontal arrows in Diagram (3.12) are injective. In this way we get a commutative diagram

$$\begin{array}{ccc} H^0_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}];\mathbb{Z}/2(n)) & \longrightarrow & H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}];\mathbb{Z}_2^\wedge(n)) \\ \downarrow{=} & & \downarrow \\ H^0_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}];\mathbb{Z}/2(n)) & \xrightarrow{\partial} & H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}];\mathbb{Z}/2(n)) \cong \langle -1,2\rangle \otimes \mathbb{Z}/2(n-1) \end{array} \tag{3.13}$$

which implies that $H^0_{\text{ét}}(\mathbb{Z}[\frac{1}{2}]; \mathbb{Z}/2(n))$ maps isomorphically to the torsion subgroup of $H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2}]; \mathbb{Z}_2^{\wedge}(n))$. On the other hand, the diagrams (3.9), (3.10), (3.12), (3.13) show that the generator of $H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2}]; \mathbb{Z}_2^{\wedge}(n))_2$ maps to the element $(-1) \otimes \xi_2^{\otimes(n-1)}$ in $\langle -1, 2\rangle \otimes \mathbb{Z}/2(n-1)$. ∎

LEMMA 3.4. *The elements e_1 and e_2 are sent by the transfer map Tr_{E_2/E_1} to the same nontorsion element $e \in H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2}]; \mathbb{Z}_2^{\wedge}(n))$ whose image $\tilde{e}$ in the $\mathbb{Z}_2^{\wedge}$-module $H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2}]; \mathbb{Z}_2^{\wedge}(n))/\text{torsion} \cong \mathbb{Z}_2^{\wedge}$ is a generator. In addition, the element e is sent to $2 \otimes \xi_2^{\otimes(n-1)}$ via the map ϕ of Lemma* 3.3.

Proof. Since the transfer map Tr_{E_2/E_1} commutes with the complex conjugation, we have:

$$Tr_{E_2/E_1}(e_2) = Tr_{E_2/E_1}(e_1^{\tau}) = (Tr_{E_2/E_1}(e_1))^{\tau} = Tr_{E_2/E_1}(e_1).$$

Let us put $e = Tr_{E_2/E_1}(e_1) \in \mathscr{C}^{\text{ét}}(n-1)$. First we prove that e is nontorsion in the group $H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2}]; \mathbb{Z}_2^{\wedge}(n))$. The composition

$$\begin{aligned} H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}]; \mathbb{Z}_2^{\wedge}(n))/\text{torsion} &\xrightarrow{\chi} H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}, i]; \mathbb{Z}_2^{\wedge}(n))/\text{torsion} \\ &\xrightarrow{Tr_{E_2/E_1}} H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}]; \mathbb{Z}_2^{\wedge}(n))/\text{torsion}, \end{aligned}$$

where χ is induced by the inclusion $\mathbb{Z}[\frac{1}{2}] \hookrightarrow \mathbb{Z}[\frac{1}{2}, i]$, is multiplication by 2 because of the basic property of the transfer. Let g be a generator of the group $H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2}]; \mathbb{Z}_2^{\wedge}(n))/\text{torsion}$. Then $\chi(g) = a\,\tilde{e}_1$ for some $a \in \mathbb{Z}_2^{\wedge}$, where $\tilde{e}_1$ denotes the image of e_1 in the group $H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2}, i]; \mathbb{Z}_2^{\wedge}(n))/\text{torsion}$. Hence, $2g = a\, Tr_{E_2/E_1}(\tilde{e}_1)$ and we see that $a = 2^i u$, where $i = 0$ or 1 and $u \in (\mathbb{Z}_2^{\wedge})^{\times}$. It follows that the element $Tr_{E_2/E_1}(e_1) = e$ is nontorsion. In order to check that $\tilde{e}$ generates $H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2}]; \mathbb{Z}_2^{\wedge}(n))/\text{torsion}$ we consider the image of e in the group

$$H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}]; \mathbb{Z}_2^{\wedge}(n))/2 \cong H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}]; \mathbb{Z}/2(n)) \cong \langle -1, 2\rangle \otimes \mathbb{Z}/2(n-1)$$

(see Lemma 3.1). It then follows by the projection formula in étale cohomology that the image of e equals

$$Tr_{E_2/E_1}(\alpha_2^{\text{ét}}((1-i) \otimes \xi_2^{\otimes(n-1)})) = 2 \otimes \xi_2^{\otimes(n-1)} \in \langle -1, 2\rangle \otimes \mathbb{Z}/2(n-1),$$

where $\xi_2 = -1$. By Lemma 3.3, the image of the generator of the module $H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2}]; \mathbb{Z}_2^{\wedge}(n))_2$ is equal to $(-1) \otimes \xi_2^{\otimes(n-1)}$. Hence, the element $\tilde{e}$ must be a generator of the $\mathbb{Z}_2^{\wedge}$-module $H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2}]; \mathbb{Z}_2^{\wedge}(n))/\text{torsion}$. ∎

THEOREM 3.5. *For any odd integer n, there is a natural isomorphism of* $\mathbb{Z}_2^\wedge$*-modules*

$$H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}]; \mathbb{Z}_2^\wedge(n)) \cong \mathscr{C}^{\text{ét}}(n-1) \oplus \mathbb{Z}/2.$$

Moreover, the $\mathbb{Z}_2^\wedge$*-module* $\mathscr{C}^{\text{ét}}(n-1)$ *is cyclic generated by the element* $e \in H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2}]; \mathbb{Z}_2^\wedge(n))$ *defined in Lemma* 3.4.

Proof. By Theorem 1(i) of [K] and Table 1 of [W2], for n odd, we have the isomorphism

$$H^1_{\text{ét}}(\mathbb{Z}[\tfrac{1}{2}]; \mathbb{Z}_2^\wedge(n)) \cong \mathbb{Z}_2^\wedge \oplus \mathbb{Z}/2.$$

Hence the theorem follows from Lemmas 3.3 and 3.4. ∎

4. RELATIVE ÉTALE K-THEORY AND COHOMOLOGY

Our next goal will be to compare 2-adic cyclotomic elements in K-theory and étale cohomology. Because the 2-adic cohomological dimension of fields with real embeddings is infinite, we need to introduce relative K-theory and cohomology.

Let $l \geqslant 2$ be any prime number and $R = \mathbb{Z}[\frac{1}{l}]$. Let $W \to Z$ be a map of two schemes over R.

DEFINITION 4.1. (a) Let $\hat{\mathbf{K}}^{\text{ét}}(W)$ and $\hat{\mathbf{K}}^{\text{ét}}(Z)$ be the Dwyer–Friedlander spectra defined with respect to the prime l, (cf. [DF], p. 256.) We define the relative étale K-theory spectrum $\hat{\mathbf{K}}^{\text{ét}}(Z, W, f)$ to be the homotopy fiber of the map of spectra $f^*\colon \hat{\mathbf{K}}^{\text{ét}}(Z) \to \hat{\mathbf{K}}^{\text{ét}}(W)$. Whenever the map f is fixed the fiber will be denoted by $\hat{\mathbf{K}}^{\text{ét}}(Z, W)$.

(b) We put $\mathbf{K}(W)$ and $\mathbf{K}(Z)$ for the Quillen K-theory spectra of W and Z. In the usual manner we define the relative K-theory spectrum $\mathbf{K}(Z, W, f)$ to be the homotopy fiber of the map of spectra $f^*\colon \mathbf{K}(Z) \to \mathbf{K}(W)$. Whenever the map f is fixed the fiber will be denoted by $\mathbf{K}(Z, W)$.

(c) We denote by $\hat{\mathscr{K}}^{\text{ét}}$ ($\mathscr{K}$, respectively) the 0th space of the Dwyer–Friedlander K-theory spectrum (the Quillen K-theory spectrum, resp.).

By definition the relative étale K-theory groups (relative étale K-theory groups with $\mathbb{Z}/l^k$ coefficients) are the homotopy groups (homotopy groups with $\mathbb{Z}/l^k$ coefficients, respectively) of the relative étale K-theory space. We use the following notation

$\hat{K}_n^{\text{ét}}(W)$	is the nth étale K-theory group of a scheme W,
$K_n^{\text{ét}}(W; \mathbb{Z}/l^k)$	is the nth étale K-group of a scheme W with coefficients $\mathbb{Z}/l^k$,
$\hat{K}_n^{\text{ét}}(Z, W)$	is the nth relative étale K-group,
$K_n^{\text{ét}}(Z, W; \mathbb{Z}/l^k)$	is the nth relative K-group with coefficients $\mathbb{Z}/l^k$.

The Quillen K-theory groups are denoted analogously. Any projective system of étale sheaves $(\mathscr{F}_i)$ on Z gives a pro-space of coefficient systems $\mathscr{M}_i$ on $Z_{\text{ét}}$. We denote $(f^*\mathscr{M}_i)$ the pro-space of coefficient systems on W corresponding to the projective system of étale sheaves $(f^*\mathscr{F}_i)$. According to [DF], p. 251 and [J], p. 223, $H^n_{\text{cont}}(Z; (\mathscr{F}_i)) = \pi_0 \operatorname{Hom}_t(Z_{\text{ét}}; K((\mathscr{M}_i), n))_R$.

Definition 4.2. We define relative $\operatorname{Hom}_t((Z, W)_{\text{ét}}, ; K((\mathscr{M}_i), n)))_R$ as the homotopy fiber of the natural map of spaces

$$\operatorname{Hom}_t(Z_{\text{ét}}; K((\mathscr{M}_i), n)))_R \to \operatorname{Hom}_t(W_{\text{ét}}; K((f^*\mathscr{M}_i), n)))_R.$$

For the description of the functor $\operatorname{Hom}_t(\ ,\ )_R$ see [DF], Definition 2.2, p. 249.

Definition 4.3. Assuming notation from [DF] let us define relative $\operatorname{Hom}_l(Z, W; B\mathscr{G}l_*(S^n))_R$ as the homotopy fiber of the natural map of spaces

$$\operatorname{Hom}_l(Z; B\overline{\mathscr{G}l_*}(S^n))_R \to \operatorname{Hom}_l(W; B\overline{\mathscr{G}l_*}(S^n))_R.$$

For the description of the functor $\operatorname{Hom}_l(\ ,\ )_R$ see [DF], Definition 2.3, p. 250.

We have relative versions of Proposition 4.5, 5.1 and 5.2 of [DF].

Proposition 4.4. *Let W and Z be two connected schemes over R with finite $\mathbb{Z}/l$-cohomological dimension $\leqslant d$. Then for any $n \geqslant 1$ there is a natural homotopy class of maps of spaces*

$$\operatorname{Hom}_l(Z, W; BGl_n)_R \to \hat{\mathscr{K}}^{\text{ét}}(Z, W)$$

which is a $(2n-d)$-equivalence. In addition, for any $n \geqslant 1$ there is a natural homotopy class of maps of spaces

$$(\hat{\mathbf{K}}^{\text{ét}}(Z, W))_n \to \operatorname{Hom}_l(Z, W; B\overline{\mathscr{G}l_*}(S^n))_R$$

such that the map for $n=0$

$$\hat{\mathscr{K}}^{\text{ét}}(Z, W) \to \Omega^n \operatorname{Hom}_l(Z, W; B\overline{\mathscr{G}l_*}(S^n))_R$$

is a weak equivalence on the connected component of the base point.

PROPOSITION 4.5. *Let* W *and* Z *be two connected schemes over* R *with finite* $\mathbb{Z}/l$*-cohomological dimension. Then there are two natural, strongly convergent fourth-quadrant spectral sequences*

$$E_2^{p,-q} = H^p_{\text{cont}}\left(Z, W; \mathbb{Z}_l^\wedge\left(\frac{q}{2}\right)\right) \Rightarrow \hat{K}^{\text{ét}}_{q-p}(Z, W),$$

$$E_2^{p,-q} = H^p_{\text{ét}}\left(Z, W; \mathbb{Z}/l^k\left(\frac{q}{2}\right)\right) \Rightarrow K^{\text{ét}}_{q-p}(Z, W; \mathbb{Z}/l^k).$$

5. COMPARISON OF 2-ADIC CYCLOTOMIC ELEMENTS IN K-THEORY AND ÉTALE COHOMOLOGY OF $\mathbb{Z}[\frac{1}{2}]$

Now we apply the results of Section 4 for $l=2$. We set the following notation:

- F is a number field with r_1 real embeddings and $2r_2$ complex embeddings,
- $\mathcal{O}_F$ is the ring of integers of F,
- $\tilde{F} = F(i)$,
- $X = spec\ \mathcal{O}_F$,
- $\tilde{X} = spec\ \mathcal{O}_{\tilde{F}}$,
- $Y = spec \bigoplus_{i=1}^{r_1} \mathbb{R}$,
- $\tilde{Y} = spec \bigoplus_{i=1}^{r_1} \mathbb{C}$.

Let us consider natural maps $F \to \bigoplus_{i=1}^{r_1} \mathbb{R}$, $\tilde{F} \to \bigoplus_{i=1}^{r_1} \mathbb{C}$, $Y \to X$ and $\tilde{Y} \to \tilde{X}$. Since the Dwyer–Friedlander spectra from Definition 4.1 are Ω-spectra, we have natural homotopy fibrations

$$\hat{\mathscr{K}}^{\text{ét}}(X, Y) \to \hat{\mathscr{K}}^{\text{ét}}(X) \to \hat{\mathscr{K}}^{\text{ét}}(Y)$$

$$\hat{\mathscr{K}}^{\text{ét}}(\tilde{X}, \tilde{Y}) \to \hat{\mathscr{K}}^{\text{ét}}(\tilde{X}) \to \hat{\mathscr{K}}^{\text{ét}}(\tilde{Y}).$$

Remark 5.1. Observe that X and Y have infinite $\mathbb{Z}/2$ cohomological dimension. However spaces $\tilde{X}$ and $\tilde{Y}$ have $\mathbb{Z}/2$ cohomological dimension equal to 2. Hence, Proposition 4.5 gives the following spectral sequences

$$E_2^{p,-q} = H^p_{\text{cont}}\left(\tilde{X}, \tilde{Y}; \mathbb{Z}_2^\wedge\left(\frac{q}{2}\right)\right) \Rightarrow \hat{K}^{\text{ét}}_{q-p}(\tilde{X}, \tilde{Y})$$

$$E_2^{p,-q} = H^p_{\text{ét}}\left(\tilde{X}, \tilde{Y}; \mathbb{Z}/2^k\left(\frac{q}{2}\right)\right) \Rightarrow K^{\text{ét}}_{q-p}(\tilde{X}, \tilde{Y}; \mathbb{Z}/2^k).$$

LEMMA 5.2. *The Dwyer-Friedlander spectral sequences from Remark* 5.1 *give the following isomorphisms for all* $j = 1, 2$ *and all* i *such that* $2i - j > 1$:

$$\hat{K}^{\text{ét}}_{2i-j}(\tilde{X}, \tilde{Y}) \cong H^j_{\text{ét}}(\tilde{X}, \tilde{Y}; \mathbb{Z}_2^\wedge(i)),$$
$$K^{\text{ét}}_{2i-j}(\tilde{X}, \tilde{Y}; \mathbb{Z}/2^k) \cong H^j_{\text{ét}}(\tilde{X}, \tilde{Y}; \mathbb{Z}/2^k(i)).$$

Proof. A straightforward computation of spectral sequences from Remark 5.1 proves our lemma for $j = 1$ with both finite and infinite coefficients and for $j = 2$ in the case of infinite coefficients. It remains to explain the case $j = 2$ with finite coefficients. In this case the spectral sequence gives the following exact sequence.

$$0 \to H^2_{\text{ét}}(\tilde{X}, \tilde{Y}; \mathbb{Z}/2^k(i)) \to K^{\text{ét}}_{2i-2}(\tilde{X}, \tilde{Y}; \mathbb{Z}/2^k) \to H^0_{\text{ét}}(\tilde{X}, \tilde{Y}; \mathbb{Z}/2^k(i)) \to 0.$$

On the other hand we have the long exact sequence in étale cohomology

$$0 \to H^0_{\text{ét}}(\tilde{X}, \tilde{Y}; \mathbb{Z}/2^k(i-1)) \to H^0_{\text{ét}}(\tilde{X}; \mathbb{Z}/2^k(i-1))$$
$$\xrightarrow{f^*} H^0_{\text{ét}}(\tilde{Y}; \mathbb{Z}/2^k(i-1)) \to,$$

where the map f^* is induced by $f\colon \tilde{Y} \to \tilde{X}$. The map f^* is an injection, hence $H^0_{\text{ét}}(\tilde{X}, \tilde{Y}; \mathbb{Z}/2^k(i-1)) = 0$, so we have the isomorphism

$$K^{\text{ét}}_{2i-2}(\tilde{X}, \tilde{Y}; \mathbb{Z}/2^k) \cong H^2_{\text{ét}}(\tilde{X}, \tilde{Y}; \mathbb{Z}/2^k(i)). \quad \blacksquare$$

Remark 5.3. The group $\mathbb{Z}/2 = \langle \tau \rangle$, where τ denotes the complex conjugation, acts on the space $\hat{\mathcal{K}}^{\text{ét}}(\tilde{X}, \tilde{Y})$. We put $\hat{\mathcal{K}}^{\text{ét}}(\tilde{X}, \tilde{Y})^{h\mathbb{Z}/2}$ to be the connected component of the base point of the homotopy fixed points space $\mathrm{Hom}_{\mathbb{Z}/2}(E\mathbb{Z}/2, \hat{\mathcal{K}}^{\text{ét}}(\tilde{X}, \tilde{Y}))$ for this action. For the homotopy fixed points space we have two strongly convergent spectral sequences [BK]:

$$E_2^{p,-q} = H^p(\mathbb{Z}/2; \hat{K}^{\text{ét}}_q(\tilde{X}, \tilde{Y})) \Rightarrow \pi_{q-p}(\hat{\mathcal{K}}^{\text{ét}}(\tilde{X}, \tilde{Y})^{h\mathbb{Z}/2})$$
$$E_2^{p,-q} = H^p(\mathbb{Z}/2; K^{\text{ét}}_q(\tilde{X}, \tilde{Y}; \mathbb{Z}/2^k)) \Rightarrow \pi_{q-p}(\hat{\mathcal{K}}^{\text{ét}}(\tilde{X}, \tilde{Y})^{h\mathbb{Z}/2}; \mathbb{Z}/2^k).$$

Remark 5.4. On the other hand we have two Hochschild-Serre spectral sequences:

$$E_2^{r,s} = H^r(\mathbb{Z}/2; H^s_{\mathrm{cont}}(\tilde{X}, \tilde{Y}; \mathbb{Z}_2^\wedge(i))) \Rightarrow H^{r+s}_{\mathrm{cont}}(X, Y; \mathbb{Z}_2^\wedge(i))$$
$$E_2^{r,s} = H^r(\mathbb{Z}/2; H^s_{\text{ét}}(\tilde{X}, \tilde{Y}; \mathbb{Z}/2^k(i))) \Rightarrow H^{r+s}_{\text{ét}}(X, Y; \mathbb{Z}/2^k(i)).$$

THEOREM 5.5. *For any positive integer n, there exist natural maps*

$$K^{\text{ét}}_{2n-1}(X, Y; \mathbb{Z}/2^k) \to H^1_{\text{ét}}(X, Y; \mathbb{Z}/2^k(n))$$
$$\hat{K}^{\text{ét}}_{2n-1}(X, Y) \to H^1_{\mathrm{cont}}(X, Y; \mathbb{Z}_2^\wedge(n)).$$

Composing the above maps with the Dwyer–Friedlander map we get natural homomorphisms:

$$c_{n,1}\colon K_{2n-1}(X, Y; \mathbb{Z}/2^k) \to H^1_{\text{ét}}(X, Y; \mathbb{Z}/2^k(n))$$
$$K_{2n-1}(X, Y) \to H^1_{\mathrm{cont}}(X, Y; \mathbb{Z}_2^\wedge(n)).$$

Proof. Since the argument for both maps is identical, we consider the case of finite cofficients. We have the natural map

$$K^{\text{ét}}_{2n-1}(X, Y; \mathbb{Z}/2^k) = \pi_{2n-1}(\hat{\mathscr{K}}^{\text{ét}}(X, Y); \mathbb{Z}/2^k)$$
$$\to \pi_{2n-1}(\hat{\mathscr{K}}^{\text{ét}}(\tilde{X}, \tilde{Y})^{h\mathbb{Z}/2}; \mathbb{Z}/2^k).$$

The edge homomorphism of the second spectral sequence of Remark 5.3 and Lemma 5.2 give the natural map

$$\pi_{2n-1}(\hat{\mathscr{K}}^{\text{ét}}(\tilde{X}, \tilde{Y})^{h\mathbb{Z}/2}; \mathbb{Z}/2^k) \to H^0(\mathbb{Z}/2; K^{\text{ét}}_{2n-1}(\tilde{X}, \tilde{Y}; \mathbb{Z}/2^k))$$
$$\cong H^0(\mathbb{Z}/2; H^1_{\text{ét}}(\tilde{X}, \tilde{Y}; \mathbb{Z}/2^k(n))).$$

Computing the second spectral sequence of Remark 5.4 we get the isomorphism

$$H^0(\mathbb{Z}/2; H^1_{\text{ét}}(\tilde{X}, \tilde{Y}; \mathbb{Z}/2^k(n))) \cong H^1_{\text{ét}}(X, Y; \mathbb{Z}/2^k(n)). \quad \blacksquare$$

LEMMA 5.6. *The first map of Theorem 5.5 is compatible with the transfer maps in étale K-theory and cohomology which goes from level* $(\tilde{X}, \tilde{Y})$ *down to level* (X, Y).

Proof. Consider the following diagram

$$\begin{array}{ccc}
K^{\text{ét}}_{2n-1}(\tilde{X}, \tilde{Y}; \mathbb{Z}/2^k) & \longrightarrow & H^1_{\text{ét}}(\tilde{X}, \tilde{Y}; \mathbb{Z}/2^k(n)) \\
\downarrow{\scriptstyle Tr} & & \downarrow{\scriptstyle Tr} \\
K^{\text{ét}}_{2n-1}(X, Y; \mathbb{Z}/2^k) & \longrightarrow & H^1_{\text{ét}}(X, Y; \mathbb{Z}/2^k(n)) \\
\downarrow & & \downarrow{\scriptstyle \cong} \\
\pi_{2n-1}(\hat{\mathscr{K}}^{\text{ét}}(\tilde{X}, \tilde{Y})^{h\mathbb{Z}/2}; \mathbb{Z}/2^k) & \longrightarrow & H^0(\mathbb{Z}/2; H^1_{\text{ét}}(\tilde{X}, \tilde{Y}; \mathbb{Z}/2^k(n))) \\
\downarrow & & \downarrow \\
K^{\text{ét}}_{2n-1}(\tilde{X}, \tilde{Y}; \mathbb{Z}/2^k) & \longrightarrow & H^1_{\text{ét}}(\tilde{X}, \tilde{Y}; \mathbb{Z}/2^k(n)).
\end{array} \tag{5.1}$$

We must prove that the upper square of the diagram commutes. The commutativity of the lowest square follows from the morphism of the [BK] spectral sequences for the group $\mathbb{Z}/2$ and for the trivial group. The definition of the map from Theorem 5.5 gives the commutativity of the middle square. Note that the composition of the left (respectively, right) vertical arrows is just the action of $1+\tau$. Hence, the large outer rectangle commutes. In addition, the right middle and lower vertical arrows are injections. It implies the commutativity of the upper square. ∎

Consider the long exact sequences

$$\to K_{2n}(Y; \mathbb{Z}_2^\wedge) \to K_{2n-1}(X, Y; \mathbb{Z}_2^\wedge) \xrightarrow{j} K_{2n-1}(X; \mathbb{Z}_2^\wedge)$$
$$\xrightarrow{i} K_{2n-1}(Y; \mathbb{Z}_2^\wedge) \to$$
$$\to H^0_{\text{cont}}(Y; \mathbb{Z}_2^\wedge(n)) \to H^1_{\text{cont}}(X, Y; \mathbb{Z}_2^\wedge(n)) \xrightarrow{j_{\text{ét}}} H^1_{\text{cont}}(X; \mathbb{Z}_2^\wedge(n))$$
$$\xrightarrow{i_{\text{ét}}} H^1_{\text{cont}}(Y; \mathbb{Z}_2^\wedge(n)) \to$$

which are obtained by taking inverse limits of the corresponding long exact sequences with finite coefficients. Exactness is preserved since the groups with finite coefficients are finite.

From now on, let us consider $F = \mathbb{Q}$ and $\mathcal{O}_F = \mathbb{Z}$. Hence, for the rest of this section we set the following notation.

$X \quad = spec\ \mathbb{Z}[\frac{1}{2}]$

$\tilde{X} \quad = spec\ \mathbb{Z}[\frac{1}{2}, i]$,

$Y \quad = spec\ \mathbb{R}$,

$\tilde{Y} \quad = spec\ \mathbb{C}$

$c_{n,1} \quad : K_{2n-1}(X, Y; \mathbb{Z}_2^\wedge) \to H^1_{\text{cont}}(X, Y; \mathbb{Z}_2^\wedge(n))$ denotes the map induced by the third homomorphism from Theorem 5.5 upon taking inverse limit on coefficients.

Consider the following commutative diagram in which the vertical sequences are exact, Tr denotes transfer maps and the horizontal arrows are Dwyer–Friedlander maps and the map of Theorem 5.5.

$$
\begin{tikzcd}[column sep=small]
 & K_{2n}(Y;\hat{\mathbb{Z}_2}) \arrow[dd] & & H^0_{\text{cont}}(Y;\hat{\mathbb{Z}_2}(n)) \arrow[dd] \\
K_{2n}(\tilde{Y};\hat{\mathbb{Z}_2}) \arrow[dd] & & & \\
 & K_{2n-1}(X,Y;\hat{\mathbb{Z}_2}) \arrow[rr, "c_{n,1}"] \arrow[dd, "j"] & & H^1_{\text{cont}}(X,Y;\hat{\mathbb{Z}_2}(n)) \arrow[dd, "j_{\acute{e}t}"] \\
K_{2n-1}(\tilde{X},\tilde{Y};\hat{\mathbb{Z}_2}) \arrow[ur, "Tr"] \arrow[rr] \arrow[dd, "\tilde{j}"] & & H^1_{\text{cont}}(\tilde{X},\tilde{Y};\hat{\mathbb{Z}_2}(n)) \arrow[ur, "Tr"] \arrow[dd, "\tilde{j}_{\acute{e}t}"] & \\
 & K_{2n-1}(X;\hat{\mathbb{Z}_2}) \arrow[dd, "i"] & & H^1_{\text{cont}}(X;\hat{\mathbb{Z}_2}(n)) \arrow[dd, "i_{\acute{e}t}"] \\
K_{2n-1}(\tilde{X};\hat{\mathbb{Z}_2}) \arrow[ur, "Tr"] \arrow[rr] \arrow[dd, "\tilde{i}"] & & H^1_{\text{cont}}(\tilde{X};\hat{\mathbb{Z}_2}(n)) \arrow[ur, "Tr"] \arrow[dd, "\tilde{i}_{\acute{e}t}"] & \\
 & K_{2n-1}(Y;\hat{\mathbb{Z}_2}) & & H^1_{\text{cont}}(Y;\hat{\mathbb{Z}_2}(n)) \\
K_{2n-1}(\tilde{Y};\hat{\mathbb{Z}_2}) \arrow[ur, "Tr"] \arrow[rr] & & H^1_{\text{cont}}(\tilde{Y};\hat{\mathbb{Z}_2}(n)) \arrow[ur, "Tr"] &
\end{tikzcd}
\tag{5.2}
$$

LEMMA 5.7. (a) *The map $\tilde{j}$ is surjective with kernel $\mathbb{Z}_2^{\wedge}$ for all $n > 1$.*
(b) *The kernel of the map j is trivial if n is odd.*

Proof. (a) follows immediately by [Su] p. 317, the relative long exact sequences in K-theory because $K_{2n}(\tilde{Y};\mathbb{Z}_2^{\wedge}) \cong \mathbb{Z}_2^{\wedge}$, $K_{2n-1}(\tilde{Y};\mathbb{Z}_2^{\wedge}) = K_{2n+1}(\tilde{Y};\mathbb{Z}_2^{\wedge}) = 0$ and since the group $K_{2n}(\tilde{X})$ is finite by [Q2].

In order to prove (b) observe that $K_{2n}(Y;\mathbb{Z}_2^{\wedge}) \cong \mathbb{Z}/2$ and by Proposition 5.5 of [ABG] the map

$$K_{2n}(X;\mathbb{Z}_2^{\wedge}) \to K_{2n}(Y;\mathbb{Z}_2^{\wedge}) \cong \mathbb{Z}/2$$

is surjective. ∎

LEMMA 5.8. *Assume that n is odd. The group $\mathscr{C}(n-1) \subseteq K_{2n-1}(X;\mathbb{Z}_2^{\wedge})$ of 2-adic cyclotomic elements is the image of a uniquely determined subgroup of $K_{2n-1}(X, Y;\mathbb{Z}_2^{\wedge})$ under the map j.*

Proof. Observe that by Definition 1.5 the group $\mathscr{C}(n-1)$ is the image of the corresponding group $\tilde{\mathscr{C}}(n-1) \subset K_{2n-1}(\tilde{X};\mathbb{Z}_2^{\wedge})$ via the transfer map from Diagram (5.2). By [Su] p. 317 we get immediately $K_{2n-1}(\tilde{Y};\mathbb{Z}_2^{\wedge}) = 0$. Hence, by the commutativity of the lowest square of the left face of Diagram (5.2) we get

$$i(\mathscr{C}(n-1)) = i(Tr(\tilde{\mathscr{C}}(n-1))) = Tr(\tilde{i}(\tilde{\mathscr{C}}(n-1))) = 0.$$

The claim follows by Lemma 5.7(b). ∎

LEMMA 5.9. *Assume that n is odd.*

(a) *The map $j_{\text{ét}}$ is injective.*

(b) *The group $\mathscr{C}^{\text{ét}}(n-1) \subseteq H^1_{\text{cont}}(X; \mathbb{Z}_2^\wedge(n))$ of 2-adic cyclotomic elements is the image of a uniquely determined subgroup of $H^1_{\text{cont}}(X, Y; \mathbb{Z}_2^\wedge(n))$ under the map $j_{\text{ét}}$.*

Proof. (a) follows because $H^0_{\text{cont}}(Y; \mathbb{Z}_2^\wedge(n)) = 0$, since n is odd. In order to prove (b) observe that by Definition 2.2 the group $\mathscr{C}^{\text{ét}}(n-1)$ is the image of the corresponding group $\widetilde{\mathscr{C}}^{\text{ét}}(n-1) \subset H^1_{\text{cont}}(\tilde{X}; \mathbb{Z}_2^\wedge(n))$ via the transfer map from Diagram (5.2). Since $H^1_{\text{cont}}(\tilde{Y}; \mathbb{Z}_2^\wedge(n)) = 0$, by the commutativity of the lowest square of the right face of Diagram (5.2) we get

$$i_{\text{ét}}(\mathscr{C}^{\text{ét}}(n-1)) = i_{\text{ét}}(Tr(\widetilde{\mathscr{C}}^{\text{ét}}(n-1))) = Tr(\tilde{i}_{\text{ét}}(\widetilde{\mathscr{C}}^{\text{ét}}(n-1))) = 0.$$

The claim (b) follows now by (a). ∎

Due to Lemma 5.8 and Lemma 5.9 we can consider the group $\mathscr{C}(n-1)$ (the group $\mathscr{C}^{\text{ét}}(n-1)$, respectively) as a subgroup of $K_{2n-1}(X, Y; \mathbb{Z}_2^\wedge)$ (of $H^1_{\text{cont}}(X, Y; \mathbb{Z}_2^\wedge(n))$, resp.).

THEOREM 5.10. *Let n be an odd natural number. Then the map*

$$c_{n,1}\colon K_{2n-1}(X, Y; \mathbb{Z}_2^\wedge) \to H^1_{\text{ét}}(X, Y; \mathbb{Z}_2^\wedge(n))$$

sends the group of 2-adic cyclotomic elements $\mathscr{C}(n-1)$ in K-theory onto the group $\mathscr{C}^{\text{ét}}(n-1)$.

Proof. This follows by an easy chase in Diagram (5.2) using Lemmas 2.4, 5.7, 5.8, and 5.9. ∎

LEMMA 5.11. *For any odd integer n, there is a natural isomorphism of $\mathbb{Z}_2^\wedge$-modules*

$$K_{2n-1}(X; \mathbb{Z}_2^\wedge) \cong K_{2n-1}(X, Y; \mathbb{Z}_2^\wedge) \oplus K_{2n-1}(Y; \mathbb{Z}_2^\wedge)$$

$$H^1_{\text{cont}}(X; \mathbb{Z}_2^\wedge(n)) \cong H^1_{\text{cont}}(X, Y; \mathbb{Z}_2^\wedge(n)) \oplus H^1_{\text{cont}}(Y; \mathbb{Z}_2^\wedge(n))$$

Moreover,

$$K_{2n-1}(X; \mathbb{Z}_2^\wedge)_{tors} \cong K_{2n-1}(Y; \mathbb{Z}_2^\wedge)$$

$$H^1_{\text{cont}}(X; \mathbb{Z}_2^\wedge(n))_{tors} \cong H^1_{\text{cont}}(Y; \mathbb{Z}_2^\wedge(n)).$$

Proof. By Lemma 5.7(b) the long exact sequence for relative K-theory reduces to the following exact sequence.

$$0 \to K_{2n-1}(X, Y; \mathbb{Z}_2^\wedge) \xrightarrow{j} K_{2n-1}(X; \mathbb{Z}_2^\wedge) \xrightarrow{i} K_{2n-1}(Y; \mathbb{Z}_2^\wedge) \to$$

We show that the map i is split surjective. We know by [Su], Theorem 4.9, p. 317 that

$$K_{2n-1}(Y; \mathbb{Z}_2^\wedge) = \begin{cases} 0 & \text{if} \quad n \equiv 3 \bmod 4 \\ \mathbb{Z}/2 & \text{if} \quad n \equiv 1 \bmod 4. \end{cases}$$

The map $i: K_{2n-1}(X; \mathbb{Z}_2^\wedge) \to K_{2n-1}(Y; \mathbb{Z}_2^\wedge)$ can be identified with the natural map

$$i: K_{2n-1}(\mathbb{Z}[\tfrac{1}{2}]) \otimes \mathbb{Z}_2^\wedge \to \pi_{2n-1}(BO) \otimes \mathbb{Z}_2^\wedge$$

which appears in the long exact sequence of homotopy groups

$$\begin{gathered} \to K_{2n}(\mathbb{Z}[\tfrac{1}{2}]) \otimes \mathbb{Z}_2^\wedge \xrightarrow{i} \pi_{2n}(BO) \otimes \mathbb{Z}_2^\wedge \to \pi_{2n-1}(SU) \otimes \mathbb{Z}_2^\wedge \to \\ \to K_{2n-1}(\mathbb{Z}[\tfrac{1}{2}]) \otimes \mathbb{Z}_2^\wedge \to \pi_{2n-1}(BO) \otimes \mathbb{Z}_2^\wedge \to \pi_{2n-2}(SU) \otimes \mathbb{Z}_2^\wedge \to \end{gathered} \tag{5.4}$$

of the homotopy fibration

$$SU_2^\wedge \to (BGL(\mathbb{Z})^+)_2^\wedge \to BO_2^\wedge,$$

cf. [ABG], Proposition 5.5. The existence of the fibration was derived in [ABG] from the calculation of the groups $K_*(\mathbb{Z}) \otimes \mathbb{Z}_2^\wedge$ cf. [RW] Theorem 0.6. For the convenience of the reader we recall the results of the calculation after the Diagram (6.1) below. In case $n \equiv 1 \bmod 4$ the sequence (5.4) gives the short exact sequence

$$0 \to \mathbb{Z}_2^\wedge \to K_{2n-1}(\mathbb{Z}[\tfrac{1}{2}]) \otimes \mathbb{Z}_2^\wedge \xrightarrow{i} \mathbb{Z}/2 \to 0 \tag{5.5}$$

which splits, because by [RW] Theorem 0.6,

$$K_{2n-1}(\mathbb{Z}[\tfrac{1}{2}]) \otimes \mathbb{Z}_2^\wedge = \mathbb{Z}_2^\wedge \oplus \mathbb{Z}/2.$$

In case $n \equiv 3 \bmod 4$ the map j is an isomorphism. Hence, for n odd, the map

$$i: K_{2n-1}(X; \mathbb{Z}_2^\wedge) \to K_{2n-1}(Y; \mathbb{Z}_2^\wedge)$$

is split surjective.

Now we prove the second claim of this lemma which concerns cohomology. We have $H^0_{\mathrm{cont}}(Y; \mathbb{Z}_2^\wedge(n)) = 0$, because n is odd. Hence, the long exact sequence for relative cohomology gives us the following exact sequence.

$$0 \to H^1_{\mathrm{cont}}(X, Y; \mathbb{Z}_2^\wedge(n)) \xrightarrow{j_{\acute{e}t}} H^1_{\mathrm{cont}}(X; \mathbb{Z}_2^\wedge(n)) \xrightarrow{i_{\acute{e}t}} H^1_{\mathrm{cont}}(Y; \mathbb{Z}_2^\wedge(n)) \to$$

We will show that the map i in this exact sequence is surjective. Consider the commutative diagram.

$$\begin{tikzcd} H^1_{\mathrm{cont}}(X; \mathbb{Z}_2^\wedge(n)) \arrow[r, "i"] & H^1_{\mathrm{cont}}(Y; \mathbb{Z}_2^\wedge(n)) \\ H^0_{\acute{e}t}(X; \mathbb{Q}_2/\mathbb{Z}_2^\wedge(n)) \arrow[u] \arrow[r, "\cong"] & H^0_{\acute{e}t}(Y; \mathbb{Q}_2/\mathbb{Z}_2^\wedge(n)) \arrow[u, "\cong"'] \end{tikzcd} \tag{5.3}$$

The right vertical arrow in the Diagram (5.3) is an isomorphism because of Lemma 4.3 of [RW]. Hence, the left vertical arrow is injective and its image is equal to the torsion subgroup of $H^1_{\mathrm{cont}}(X; \mathbb{Z}_2^\wedge(n))$, because $H^1_{\mathrm{cont}}(X; \mathbb{Z}_2^\wedge(n))_{tors} \cong \mathbb{Z}/2$ and $H^0_{\acute{e}t}(Y; \mathbb{Q}_2/\mathbb{Z}_2^\wedge(n)) \cong \mathbb{Z}/2$. So the Lemma follows. ∎

Corollary 5.12. *For any n odd there is a natural isomorphism*

$$H^1_{\mathrm{cont}}(X, Y; \mathbb{Z}_2^\wedge(n)) \cong \mathscr{C}^{\acute{e}t}(n-1).$$

Proof. This follows immediately by Lemma 5.9 and Theorems 3.5, 5.10. ∎

We are going to investigate which part of the group $K_{2n-1}(\mathbb{Z}) \otimes \mathbb{Z}_2^\wedge$ for n odd, is described by the cyclotomic elements. We will use coinvariants of the $G_m = Gal(E_m/\mathbb{Q})$-module $C'_m \otimes \mathbb{Z}/2^m(n-1)$.

Lemma 5.13. *For $m \geqslant 2$ and n odd, the module of coinvariants*

$$[C'_m \otimes \mathbb{Z}/2^m(n-1)]_{G_m}$$

of the group G_m is generated by classes represented by $(-1) \otimes \xi_{2^m}^{\otimes(n-1)}$ and $(1-\xi_{2^m}) \otimes \xi_{2^m}^{\otimes(n-1)}$.

Proof. For any a odd, $1 < a \leqslant 2^m$ let $\sigma_a \in G_m$ denote the automorphism such that $\sigma_a(\xi_{2^m}) = \xi_{2^m}^a$. Observe that the automorphisms σ_a generate G_m. We have

$$\begin{aligned} (1-\xi_{2^m}^a) \otimes \xi_{2^m}^{\otimes(n-1)} &= [(1-\xi_{2^m})^{b^{n-1}} \otimes \xi_{2^m}^{\otimes(n-1)}]^{\sigma_a} \\ &= [(1-\xi_{2^m}) \otimes \xi_{2^m}^{\otimes(n-1)}]^{b^{n-1}} \\ &\quad \times [(1-\xi_{2^m})^{b^{n-1}} \otimes \xi_{2^m}^{\otimes(n-1)}]^{\sigma_a - 1}, \end{aligned}$$

where b is such that $ba \equiv 1 \bmod 2^m$. Note also that $\tau = \sigma_{-1}$ and we have

$$(1-\xi_{2^m})^{\tau} = 1 - \xi_{2^m}^{-1} = \xi_{2^m}^{-1}(\xi_{2^m} - 1),$$

so we get

$$-\xi_{2^m} = \left(\frac{1}{1-\xi_{2^m}}\right)^{\tau - 1}.$$

Hence, also

$$(-\xi_{2^m}) \otimes \xi_{2^m}^{\otimes (n-1)} = \left[\left(\frac{1}{1-\xi_{2^m}}\right) \otimes \xi_{2^m}^{\otimes (n-1)}\right]^{\tau - 1}$$

because n is odd, by assumption. The lemma follows by definition of the module of coinvariants and Remark 1.2. ∎

THEOREM 5.14. *For any odd integer n, there is a natural isomorphism of $\mathbb{Z}_2^{\wedge}$-modules*

$$K_{2n-1}(\mathbb{Z}) \otimes \mathbb{Z}_2^{\wedge} \cong \begin{cases} \mathscr{C}(n-1) & \text{if} \quad n \equiv 3 \bmod 4 \\ \mathscr{C}(n-1) \oplus \mathbb{Z}/2 & \text{if} \quad n \equiv 1 \bmod 4. \end{cases}$$

Moreover, the $\mathbb{Z}_2^{\wedge}$-module $\mathscr{C}(n-1)$ is isomorphic to $\mathbb{Z}_2^{\wedge}$.

Proof. We define the inverse system of groups (A_m) as follows

$$A_m = \begin{cases} [C'_m \otimes \mathbb{Z}/2^m(n-1)]_{G_m} & \text{if} \quad m \geqslant 2 \\ A_2/2A_2 & \text{if} \quad m = 1. \end{cases}$$

Note that $-1 = i^2$, so $A_1 = A_2/2A_2$ is generated by the image of the class of $(1-i) \otimes i^{\otimes (n-1)}$ in A_2. Hence, we have $A_1 \cong \mathbb{Z}/2$. We define maps $q_m \colon A_m \to A_{m-1}$ of the inverse system. Put

$$q_m = \begin{cases} N_{E_m/E_{m-1}} \circ r_m & \text{if} \quad m > 2 \\ r_1 & \text{if} \quad m = 2. \end{cases}$$

where r_1 is the obvious quotient map and r_m, for $m > 1$ is the reduction of coefficients. The map $N_{E_m/E_{m-1}}$ is the norm map. We put $A^{\wedge} = \varprojlim A_m$. Observe that $A^{\wedge}$ is a cyclic $\mathbb{Z}_2^{\wedge}$-module which is generated by $(a_m)_m$, where

$$a_m = \begin{cases} (1-\xi_{2^m}) \otimes \xi_{2^m}^{\otimes (n-1)} & \text{if} \quad m \geqslant 2 \\ q_2((1-i) \otimes i^{\otimes (n-1)}) & \text{if} \quad m = 1. \end{cases}$$

Because of the construction of the system (A_m) and the definitions of cyclotomic elements (see Definitions 2.1 and 2.2) we have the following diagram with natural maps.

$$\begin{array}{ccccc} A^{\wedge} & \xrightarrow{c} & K_{2n-1}(\mathbb{Z}[\frac{1}{2}]) \otimes \mathbb{Z}_2^{\wedge} & \xleftarrow{j} & K_{2n-1}(X, Y; \mathbb{Z}_2^{\wedge}) \\ \Big\downarrow{=} & & & & \Big\downarrow{c_{n,1}} \\ A^{\wedge} & \xrightarrow{c_{\text{ét}}} & H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2}]; \mathbb{Z}_2^{\wedge}(n)) & \xleftarrow{j_{\text{ét}}} & H^1_{\text{cont}}(X, Y; \mathbb{Z}_2^{\wedge}(n)). \end{array} \tag{5.3}$$

The maps c and $c_{\text{ét}}$ are induced by the maps $Tr_{E_m/\mathbb{Q}} \circ \alpha_m$ and $Tr_{E_m/\mathbb{Q}} \circ \alpha_m^{\text{ét}}$, which were introduced after Remark 1.4 and before Lemma 2.1, respectively. The maps j and $j_{\text{ét}}$ were introduced before Diagram (5.2). Note that $Im\, c = \mathscr{C}(n-1)$ and $Im\, c_{\text{ét}} = \mathscr{C}^{\text{ét}}(n-1)$. Observe also that, due to Lemmas 5.7, 5.8, 5.9 and Theorem 5.10 we have the identity

$$c_{n,1} \circ j^{-1} \circ c = j_{\text{ét}}^{-1} \circ c_{\text{ét}}. \tag{5.4}$$

By Lemma 3.4 and Theorem 3.5 we have

$$c_{\text{ét}}(A^{\wedge}) = \mathscr{C}^{\text{ét}}(n-1) \cong \mathbb{Z}_2^{\wedge}. \tag{5.5}$$

Since $A^{\wedge}$ is a cyclic $\mathbb{Z}_2^{\wedge}$-module, (5.5) implies that $A^{\wedge} \cong \mathbb{Z}_2^{\wedge}$. By the identity (5.4) and chasing in the Diagram (5.3) we get the following isomorphism

$$\mathscr{C}(n-1) \cong \mathscr{C}^{\text{ét}}(n-1) \tag{5.6}$$

Lemmas 5.8, 5.9, Theorem 5.10, the isomorphism (5.6), and the structure of the group $K_{2n-1}(\mathbb{Z}) \otimes \mathbb{Z}_2^{\wedge}$, described in Theorem 0.6 of [RW] (see also formulas after Diagram (6.1) below), give the first claim of the theorem. ∎

COROLLARY 5.15. *For any n odd there is a natural isomorphism*

$$K_{2n-1}(X, Y; \mathbb{Z}_2^{\wedge}) \cong \mathscr{C}(n-1).$$

Proof. This follows immediately by Lemmas 5.8, Lemma 5.11 and Theorem 5.14. ∎

6. 2-ADIC CYCLOTOMIC ELEMENTS AND PRODUCTS IN K-THEORY OF $\mathbb{Z}$

In this section we apply 2-adic cyclotomic elements in K-theory and étale cohomology to compute some of the product maps in the 2-adic K-theory of $\mathbb{Z}$. By Theorem 5.14 there exists an element $b_n \in K_{2n-1}(\mathbb{Z})$, such that $b_n \otimes 1$ is a generator of the free $\mathbb{Z}_2^\wedge$-module

$$\mathscr{C}(n-1) \subset K_{2n-1}(\mathbb{Z}) \otimes \mathbb{Z}_2^\wedge .$$

With no loss of generality, we can assume that $c_{n,1}(b_n \otimes 1) = \tilde{e}$, where $\tilde{e}$ is the generator of $\mathscr{C}^{\text{ét}}(n-1) \subseteq H^1_{\text{ét}}(\mathbb{Z}[\frac{1}{2}]; \mathbb{Z}_2^\wedge(n))$ given by Lemma 3.4. We observe that for n odd $\geqslant 3$, the element b_n is such that the image of b_n in $K_{2n-1}(\mathbb{Z})/\text{torsion} \cong \mathbb{Z}$ is an odd multiple of a generator.

THEOREM 6.1. *Assume that n and m are odd integers $\geqslant 3$.*

(a) *The product map*

$$\star : K_1(\mathbb{Z}) \otimes K_{2m-1}(\mathbb{Z}) \to K_{2m}(\mathbb{Z})$$

sends the subgroup $\langle -1 \rangle \otimes \langle b_m \rangle$ to zero.

(b) *The 2-adic product map*

$$\star : K_{2n-1}(\mathbb{Z}) \otimes K_{2m-1}(\mathbb{Z}) \to K_{2(n+m-1)}(\mathbb{Z}) \otimes \mathbb{Z}_2^\wedge$$

sends the subgroup $\langle b_n \rangle \otimes \langle b_m \rangle$ to zero.

Proof. Since we are only interested in the image of the product (a) in the 2-torsion part of $K_{2m}(\mathbb{Z})$, it is sufficient to consider the following commutative diagram

$$\begin{array}{ccc} K_1(\mathbb{Z})/4 \otimes K_{2m-1}(\mathbb{Z})/4 & \xrightarrow{\star} & K_{2m}(\mathbb{Z})_2 \\ \downarrow & & \downarrow \\ K_1(\mathbb{Z})/4 \otimes K_{2m-1}(\mathbb{Z}; \mathbb{Z}/4) & \xrightarrow{\star} & K_{2m}(\mathbb{Z}; \mathbb{Z}/4). \end{array} \tag{6.1}$$

According to Table 1 of [W2] and Theorem 0.6 of [RW],

$$K_i(\mathbb{Z}) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \text{finite odd torsion group}, & \text{if } \; i \equiv 1 \bmod 8, \\ \mathbb{Z}/2 \oplus \text{finite odd torsion group}, & \text{if } \; i \equiv 2 \bmod 8, \\ \mathbb{Z}/16 \oplus \text{finite odd torsion group}, & \text{if } \; i \equiv 3 \bmod 8, \\ \mathbb{Z} \oplus \text{finite odd torsion group}, & \text{if } \; i \equiv 5 \bmod 8, \\ \mathbb{Z}/(2(i+1)_2) \oplus \text{finite odd torsion group}, & \text{if } \; i \equiv 7 \bmod 8, \\ \text{finite odd torsion group}, & \text{if otherwise}, \end{cases}$$

where $(x)_2$ denotes the 2-part of the rational number x. In particular, $K_{2m}(\mathbb{Z}) \otimes \mathbb{Z}_2^\wedge \cong \mathbb{Z}/2$ if $m \equiv 1 \bmod 4$ and the right vertical arrow in Diagram (6.1) is a well defined embedding. Hence, to prove claim (a) we need to show that $\langle -1\rangle \otimes \langle b_m\rangle$ maps to zero in $K_{2m}(\mathbb{Z};\mathbb{Z}/4)$ via the homomorphisms in Diagram (6.1). The left vertical arrow in Diagram (6.1) maps the element $(-1)\otimes b_m$ to

$$(-1)\otimes Tr_{E_2/E_1}(c \star \beta_2^{\star(m-1)})$$

for some $c \in C_2'$. Sending it further to $K_{2m}(\mathbb{Z};\mathbb{Z}/4)$ along the lower horizontal arrow and using the projection formula and associativity of the product for K-groups with $\mathbb{Z}/4$ coefficients we find out that its image equals

$$Tr_{E_2/E_1}(((-1)\star c)\star \beta_2^{\star(m-1)}).$$

The product $\star\colon K_1(R)\otimes K_1(R)\to K_2(R)$ with integral coefficients is just the Steinberg symbol $\{\ ,\ \}$ for any commutative ring with identity R (see [L], Proposition 2.2.3). Hence, we get:

$$(-1)\star(1-i)=\{-1,1-i\}=\{i,1-i\}^2=1$$
$$(-1)\star(1+i)=\{-1,1+i\}=\{-i,1+i\}^2=1.$$

But the group C_2' is generated by $1-i$ and $1+i$, hence $(-1)\star c=1$. This proves (a) for $m\equiv 1 \bmod 4$ and there is nothing to prove if $m\equiv 3 \bmod 4$ since $K_{2m}(\mathbb{Z})_2=0$ by [W2].

In order to prove (b) consider the commutative diagram

$$\begin{array}{ccc} K_{2n-1}(\mathbb{Z})/4\otimes K_{2m-1}(\mathbb{Z})/4 & \xrightarrow{\ \star\ } & K_{2(n+m-1)}(\mathbb{Z})_2 \\ \downarrow & & \downarrow \\ K_{2n-1}(\mathbb{Z};\mathbb{Z}/4)\otimes K_{2m-1}(\mathbb{Z};\mathbb{Z}/4) & \xrightarrow{\ \star\ } & K_{2(n+m-1)}(\mathbb{Z};\mathbb{Z}/4). \end{array} \tag{6.2}$$

It is enough to show that the image of $b_n\otimes b_m$ in the upper left corner of Diagram (6.2) maps to the trivial element in the lower right corner on its journey through the diagram. Indeed, $b_n\otimes b_m$ is sent to

$$Tr_{E_2/E_1}(c_1\star \beta_2^{\star(n-1)})\otimes Tr_{E_2/E_1}(c_2\star \beta_2^{\star(m-1)})$$

via the left vertical arrow in Diagram (6.2) for some $c_1, c_2\in C_2'$. Sending the last element along the bottom horizontal arrow and using projection formula we get the element

$$Tr_{E_2/E_1}(c_1\star \beta_2^{\star(n-1)}\star c_2c_2^{\tau}\star \beta_2^{\star(m-1)}),$$

where τ denotes the complex conjugation. Since β_2 commutes with elements $c\in K_1(R)$, we obtain the following equality:

$$c_1\star \beta_2^{\star(n-1)}\star c_2c_2^{\tau}\star \beta_2^{\star(m-1)} = \pm c_1\star c_2c_2^{\tau}\star \beta_2^{\star(n+m-2)}.$$

On the other hand, we have the following equalities involving Steinberg symbols:

$$\begin{aligned}
(1-i) \star (1-i) &= \{1-i, 1-i\} = \{1-i, -1\} = \{1-i, i\}^2 = 1 \\
(1-i) \star (1+i) &= \{1-i, 1+i\} = \{1-i, i(1-i)\} \\
&= \{1-i, i\}\{1-i, 1-i\} = 1 \\
(1+i) \star (1-i) &= \{1-i, 1+i\}^{-1} = 1 \qquad (6.3) \\
(1+i) \star (1+i) &= \{1+i, 1+i\} = \{1+i, i(1-i)\} \\
&= \{1+i, i\}\{1+i, 1-i\} \\
&= \{1+i, i\} = \{1+i, -1\} = 1
\end{aligned}$$

The group C_2' is generated by $1-i$ and $1+i$ (see Example 1.3). Hence, by bilinearity of the Steinberg symbol and by (6.3) we get $c_1 \star c_2 c_2^{\tau} = \{c_1, c_2 c_2^{\tau}\} = 1$ and it follows that

$$Tr_{E_2/E_1}(c_1 \star \beta_2^{\star(n-1)} \star c_2 c_2^{\tau} \star \beta_2^{\star(m-1)}) = 1.$$

So $b_n \otimes b_m$ maps to zero in $K_{2(n+m-1)}(\mathbb{Z})_2$ via the horizontal arrow of Diagram (6.2) because the right vertical arrow is injective. ∎

Remark 6.2. We can prove claim (a) in Theorem 6.1 without even knowing the 2-torsion of the group $K_{2m}(\mathbb{Z})$. In order to see this, fix $s > 1$ such that 2^s annihilates the group $K_{2m}(\mathbb{Z})_2$. Consider the commutative diagram

$$\begin{tikzcd}
K_1(\mathbb{Z}) \otimes K_{2m-1}(\mathbb{Z}) \arrow[r, "\star"] \arrow[d] & K_{2m}(\mathbb{Z})_2 \arrow[d] \\
K_1(\mathbb{Z})/2^s \otimes K_{2m-1}(\mathbb{Z}; \mathbb{Z}/2^s) \arrow[r, "\star"] & K_{2m}(\mathbb{Z}; \mathbb{Z}/2^s),
\end{tikzcd}$$

where the right vertical arrow is injective by assumptions. Now we argue in the same way as in the proof of (a) in Theorem 6.1. The crucial point is to prove that the element $Tr_{E/\mathbb{Q}}((-1) \star u_s \star \beta_s^{\star(m-1)}) \in K_{2m}(\mathbb{Z}; \mathbb{Z}/2^s)$ is trivial for any $u_s \in C_s'$. But we know the generators of the group C_s' by Remark 1.2, so it is enough to observe the following equalities on Steinberg symbols:

$$\{-1, \xi_{2^s}\} = \{-\xi_{2^s}, \xi_{2^s}\}^{2^{s-1}} = 1 \qquad \text{and}$$

$$\{-1, 1-\xi_{2^s}^a\} = \{\xi_{2^s}^a, 1-\xi_{2^s}^a\}^{2^{s-1}} = 1.$$

Remark 6.3. Note that the result of Theorem 6.1 follows also from [ABG] Theorems 5.6 and 5.7.

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